POINTS OF BOUNDED HEIGHT ON EQUIVARIANT COMPACTIFICATIONS OF VECTOR GROUPS, II

by

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Abstract. — We prove asymptotic formulas for the number of rational points of bounded height on certain equivariant compactifications of the affine space.

Résumé. — Nous établissons un développement asymptotique du nombre de points rationnels de hauteur bornée sur certaines compactifications équivariantes de l'espace affine.

Contents

Introduction

In the last decade, there has been much interest in establishing asymptotics for the number of points of bounded height on algebraic varieties defined over number fields. Yu. Manin and V. Batyrev [**1**] have formulated conjectures describing such asymptotics in geometrical terms. These conjectures have been further refined by E. Peyre in [**8**].

More precisely, let X be a smooth projective algebraic variety defined over a number field F and $H : X(F) \to \mathbf{R}_{>0}$ an exponential height function on the

set of rational points of X defined by some metrized ample line bundle $\mathscr{L}$. One wants to relate the asymptotic behaviour of the counting function

$$N(U, \mathscr{L}, B) = \#\{x \in U(F)\,;\, H(x) \leqslant B\}$$

to geometric invariants of X, such as the cone of effective line bundles and the (anti)-canonical line bundle of X. Here, U is a sufficiently small Zariski dense open subset; its presence is made necessary by possible "accumulating subvarieties", which contain more rational points than their complement in X. If X is a Fano variety and $\mathscr{L} = K_X^{-1}$, one expects that

$$N(U, K_X^{-1}, B) \sim \frac{\Theta(X)}{(r-1)!} B(\log B)^{r-1}$$

where $r = \operatorname{rk} \operatorname{Pic}(X)$ and $\Theta(X)$ is the product of three numbers: a Tamagawa constant which measures the volume of the closure of rational points in the adelic points $\overline{X(F)} \subset X(\mathbf{A}_F)$ with respect to the metrization, a rational number defined in terms of the cone of effective divisors and the order of the non-trivial part of the Brauer group of X.

Such a description cannot hold universally (see the example by V. Batyrev and Yu. Tschinkel [**2**]), but there are two classes of algebraic varieties where it does hold: those for which the *circle method* in analytic number theory applies, and those possessing many symmetries, such as an *action* (with a dense orbit) of a linear algebraic group. The circle method is concerned with complete intersections of small degree and small codimension in projective space. They have moduli, but only few projective embeddings; the Picard group is $\mathbf{Z}$. As a reference, let us mention the papers by B. Birch [**4**] and W. Schmidt [**9**]. The other approach leads, via harmonic analysis on the adelic points of the corresponding group, to a proof of conjectured asymptotic formulas for toric varieties (see [**3**]) or for generalized flag varieties (using Langlands' work on Eisenstein's series, see [**6**]). These have Picard groups of higher ranks, but no deformations due to the *rigidity* of reductive groups.

In this paper we treat certain equivariant compactifications of vector groups. In a previous paper [**5**], we had established asymptotic formulas for blow-ups of $\mathbf{P}^2$ in any number of points *on a line*. Here we work out the case of blow-ups of a projective space $\mathbf{P}^n$ of dimension at least 3 in a smooth codimension 2 subvariety contained in a hyperplane. It should be clear to the reader that these varieties admit deformations (they are parametrized by an open subset of an appropriate Hilbert scheme).

More precisely, let $f \in \mathbf{Z}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d and $X \to \mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[x_0, \dots, x_n])$ be the blow-up of the ideal generated by (x_0, f). Suppose that the hypersurface defined by f in $\mathbf{P}^{n-1}_{\mathbf{C}}$ is smooth and let $U \simeq \mathbf{A}^n$ be the inverse image in X of $\mathbf{A}^n \subset \mathbf{P}^n$. Then, $X_{\mathbf{C}}$ is a smooth projective variety, with Picard group $\mathbf{Z}^2$ and trivial Brauer group. Moreover, $X_{\mathbf{C}}$ is an equivariant compactification of $\mathbf{G}_a^n$. There is a natural metrization on K_X^{-1} (recalled below) which allows to define the height function and the *height zeta function*

$$Z(U, K_X^{-1}, s) = \sum_{x \in U(\mathbf{Q})} H_{K_X^{-1}}(x)^{-s},$$

The series converges absolutely for $\mathrm{Re}(s) \gg 0$. Our main theorem is:

Theorem 1. — *There exists a function h which is holomorphic in the domain $\mathrm{Re}(s) > 1 - \frac{1}{n}$ such that*

$$Z(U, K_X^{-1}, s) = \frac{h(s)}{(s-1)^2} \quad \textit{and} \quad h(1) = \Theta(X) \neq 0.$$

A standard Tauberian theorem implies that X satisfies Peyre's refinement of Manin's conjecture:

Corollary 2. — *We have the following asymptotic formula:*

$$N(U, K_X^{-1}, B) \sim \Theta(X) B \log(B)$$

as B tends to infinity.

In fact, we will prove asymptotics for every $\mathscr{L}$ on X such that its class is contained in the interior of the effective cone $\Lambda_{\mathrm{eff}}(X)$. Moreover, we will prove estimates for the growth of $Z(s)$ in vertical strips in the neighbourhood of $\mathrm{Re}(s) = 1$. It is well known that this implies a more precise asymptotic expansion for the counting function $N(U, \mathscr{L}, B)$, see Theorem 4.4 and its corollary at the end of the paper.

§ 1. Geometry, heights

Let $f \in \mathbf{Z}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d with coprime coefficients and $\pi : X \to \mathbf{P}^n$ the blow-up of the ideal (x_0, f) in $\mathbf{P}^n = \operatorname{Proj}(\mathbf{Z}[x_0, \dots, x_n])$. We denote by Z_f the hypersurface defined by f in $\mathbf{P}^{n-1}$. Throughout the paper, we assume that $Z_{f,\mathbf{C}}$ is smooth, irreducible and that it doesn't contain any hyperplane. In other words, $n \geqslant 3$ and $d \geqslant 2$. The universal property of blowing up implies that the scheme X is an equivariant compactification of the additive group $\mathbf{G}_a^n = \operatorname{Spec}(\mathbf{Z}[x_1, \dots, x_n])$.

Denote by D_1 the exceptional divisor in X and by D_0 the strict transform of the divisor $x_0 = 0$ in $\mathbf{P}^n$. Let $U \simeq \mathbf{G}_a^n$ be the inverse image of $\mathbf{G}_a^n$ under π. We identify rational points in U with their image in the affine space $\mathbf{G}_a^n \subset \mathbf{P}^n$.

If $\mathbf{s} \in \mathbf{C}^2$, denote $D(\mathbf{s}) = s_0[D_0] + s_1[D_1] \in \operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}$.

The following proposition summarizes the geometric facts needed in the sequel.

Proposition 1.1. — *The classes of the divisors D_0 and D_1 form a basis of* $\operatorname{Pic}(X)$. *For $\mathbf{s} = (s_0, s_1) \in \mathbf{Z}^2$, the divisor class $D(\mathbf{s})$ is effective iff $s_0 \geqslant 0$ and $s_1 \geqslant 0$. The variety $X_{\mathbf{Q}}$ is smooth; its anticanonical line bundle has class $D(n+1, n)$.*

Proof. — See [**5**], Prop. 1.3 and Prop. 1.6 or [**7**], chap. II, § 8. □

We now define height functions on X. We denote by $\operatorname{Val}(\mathbf{Q}) = \{2, 3, \dots, \infty\}$ the set of places of $\mathbf{Q}$. If p is a prime number and $\mathbf{x} \in \mathbf{G}_a^n(\mathbf{Q}_p)$, let $\|\mathbf{x}\|_p = \max(|x_1|_p, \dots, |x_n|_p)$ and define the functions $H_{D_1,p}$ and $H_{D_0,p}$ by

$$H_{D_1,p}(\mathbf{x})^{-1} = \max\Big(\frac{1}{\max(1, \|\mathbf{x}\|_p)}, \frac{|f(\mathbf{x})|_p}{\max(1, \|\mathbf{x}\|_p)^d}\Big) \tag{1.2}$$

$$H_{D_0,p}(\mathbf{x})^{-1} = \frac{H_{D_1,p}(\mathbf{x})}{\max(1, \|\mathbf{x}\|)}. \tag{1.3}$$

At the archimedean place of $\mathbf{Q}$, define the local height functions by replacing maximums by the square root of the sum of squares. For any place v of $\mathbf{Q}$ and any $\mathbf{s} = (s_0, s_1) \in \mathbf{C}^2$, we set

$$H_v(\mathbf{s}; \mathbf{x}) = H_{D_0,v}(\mathbf{x})^{s_0} H_{D_1,v}(\mathbf{x})^{s_1}. \tag{1.4}$$

Finally, we define a global height pairing

$$H : \operatorname{Pic}(X)_{\mathbf{C}} \times \mathbf{G}_a^n(\mathbf{A}_{\mathbf{Q}}) \to \mathbf{C}^*, \quad H(\mathbf{s}; \mathbf{x}) = \prod_{v \in \operatorname{Val}(\mathbf{Q})} H_v(\mathbf{s}; \mathbf{x}_v). \tag{1.5}$$

Proposition 1.6. — *If* $\mathscr{L} \in \mathrm{Pic}(X)$*, the function* $\mathbf{x} \mapsto H(\mathscr{L};\mathbf{x})$ *on* $\mathbf{G}_a^n(\mathbf{Q})$ *is an exponential height in the sense of Weil.*

Proof. — See [**5**], (1.12), (1.13) and (2.2). □

The height zeta function is then defined by the series

$$Z(\mathbf{s}) = \sum_{\mathbf{x}\in\mathbf{G}_a^n(\mathbf{Q})} H(\mathbf{s};\mathbf{x})^{-1}. \tag{1.7}$$

It converges *a priori* for all $\mathbf{s} \in \mathbf{C}^2$ such that $D(\mathbf{s})$ is sufficiently ample, i.e. if $\mathrm{Re}(s_0 - s_1)$ and $\mathrm{Re}(s_0)$ are big enough.

Let $\psi = \prod_v \psi_v : \mathbf{G}_a(\mathbf{A}_\mathbf{Q}) \to \mathbf{C}^*$ be the standard additive character of $\mathbf{A}_\mathbf{Q}$. If $\mathbf{a} \in \mathbf{Q}^n$, we define

$$\psi_\mathbf{a}(\mathbf{x}) = \psi(\langle \mathbf{a}, \mathbf{x}\rangle).$$

We use the standard self-dual Haar measure $\mathrm{d}\mathbf{x}$ on $\mathbf{G}_a^n(\mathbf{A}_\mathbf{Q})$. For any $\mathbf{a} \in \mathbf{Q}^n$, define the Fourier transform

$$\hat{H}(\mathbf{s};\psi_\mathbf{a}) = \int_{\mathbf{G}_a^n(\mathbf{A}_\mathbf{Q})} H(\mathbf{s};\mathbf{x})^{-1}\,\mathrm{d}\mathbf{x}.$$

It is the product of the local Fourier transforms $\hat{H}_v(\mathbf{s};\psi_\mathbf{a})$.

For $\mathbf{s} \in \mathbf{C}^2$ such that both sides converge absolutely, we have the following identity:

$$Z(\mathbf{s}) = \sum_{\mathbf{a}\in\mathbf{Z}^n} \hat{H}(\mathbf{s};\psi_\mathbf{a}). \tag{1.8}$$

This is a consequence of the usual Poisson formula, see [**5**], end of § 2.

In the following sections we determine the domain of absolute convergence of the right hand side and prove that $Z(\mathbf{s})$ admits a meromorphic continuation beyond this domain.

§ 2. The local Fourier transform at the trivial character

We denote by S the minimal set of primes such that $Z_f \subset \mathbf{P}^{n-1}_\mathbf{Z}$ is smooth over $\mathrm{Spec}\,\mathbf{Z}[S^{-1}]$. Let p be a prime number.

2.1. Decomposition of the domain. — We define subsets of $\mathbf{Q}_p^n$ as follows:
- $U(0) = \mathbf{Z}_p^n$;
- if $0 \leqslant \beta < \alpha$, $U_1(\alpha, \beta)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| = p^{d\alpha-\beta}$;
- if $\alpha \geqslant 1$, $U_1(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| \leqslant p^{(d-1)\alpha}$;
- if $\alpha \geqslant 1$, $U(\alpha)$ is the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| = p^{d\alpha}$.

The local height function is constant on each of these subsets. Namely, if $\mathbf{x} \in U(0)$, $H_{D_0,p} = H_{D_1,p} = 1$. If $\mathbf{x} \in U_1(\alpha, \beta)$, $H_{D_0,p} = p^{\alpha-\beta}$ and $H_{D_1,p} = p^\beta$. On $U(\alpha)$, $H_{D_0,p} = p^\alpha$ and $H_{D_1,p} = 1$. Finally, if $\mathbf{x} \in U(\alpha)$, then $H_{D_0,p} = 1$ and $H_{D_1,p} = p^\alpha$.

2.2. Volumes. — Denote by

$$\tau_p(f) = \big(1 - \frac{1}{p}\big)\frac{\#Z_f(\mathbf{F}_p)}{p^{n-2}}.$$

The Weil conjectures proved by Deligne imply that $\tau_p(f) = 1 + O(1/p)$. In a much more elementary way, it follows from Lemma 3.9 below that $\tau_p(f)$ is bounded as p varies.

Lemma 2.3. — *For $p \notin S$, we have*

$$\operatorname{vol}(U(0)) = 1 \tag{2.3a}$$

$$\operatorname{vol}(U_1(\alpha, \beta)) = \frac{p-1}{p}\tau_p(f)p^{n\alpha-\beta} \tag{2.3b}$$

$$\operatorname{vol}(U_1(\alpha)) = \tau_p(f)p^{(n-1)\alpha} \tag{2.3c}$$

$$\operatorname{vol}(U(\alpha)) = \big(1 - p^{-n} - p^{-1}\tau_p(f)\big)p^{n\alpha}. \tag{2.3d}$$

Proof. — For $\beta \geqslant 1$, let $\Omega(\beta)$ be the set of $\mathbf{x} \in \mathbf{Z}_p^n$ such that $\|\mathbf{x}\| = 1$ and $|f(\mathbf{x})| \leqslant p^{-\beta}$. By definition,

$$\operatorname{vol}(\Omega(\beta)) = p^{-n\beta}p^{\beta-1}(p-1)\#Z_f(\mathbf{Z}/p^\beta\mathbf{Z}).$$

As Z_f is smooth of pure dimension $n-2$ over $\mathbf{Z}_p$, Hensel's lemma implies that

$$\#Z_f(\mathbf{Z}/p^\beta\mathbf{Z}) = p^{(\beta-1)(n-2)}\#Z_f(\mathbf{F}_p).$$

Consequently,

$$\operatorname{vol}(\Omega(\beta)) = (p-1)p^{-\beta-1}\frac{\#Z_f(\mathbf{F}_p)}{p^{n-2}} = \tau_p(f)p^{-\beta}.$$

As $U_1(\alpha) = p^{-\alpha}\Omega(\alpha)$, we have

$$\mathrm{vol}(U_1(\alpha)) = \tau_p(f)p^{(n-1)\alpha}.$$

Now,

$$U_1(\alpha,\beta) = p^{-\alpha}U_1(0,\beta) = p^{-\alpha}\big(\Omega(\beta) - \Omega(\beta+1)\big),$$

therefore

$$\mathrm{vol}(U_1(\alpha,\beta)) = \frac{p-1}{p}\tau_p(f)p^{n\alpha-\beta}.$$

Finally, $U(\alpha) = p^{-\alpha}(\mathbf{Z}_p^n \setminus (p\mathbf{Z}_p^n \cup \Omega(1)))$, hence

$$\mathrm{vol}(U(\alpha)) = \big(1 - p^{-n} - p^{-1}\tau_p(f)\big)p^{n\alpha}.$$

□

Proposition 2.4. — *Assume that $p \notin S$. Then,*

$$\hat{H}_p(\mathbf{s};\psi_0) = \hat{H}_{\mathbf{P}^n,p}(s_0) + \tau_p(f)\frac{p^{s_0-n} - p^{s_1-n}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)}$$

where

$$\hat{H}_{\mathbf{P}^n,p}(s_0) = \frac{1-p^{-s_0}}{1-p^{n-s_0}}$$

denotes the Fourier transform (with respect to the trivial character ψ_0) of the local height function of $\mathbf{P}^n$ for the tautological line bundle at s_0.

Proof. — By definition,

$$\begin{aligned}\hat{H}_p(\mathbf{s};\psi_0) &= \int_{\mathbf{Q}_p^n} H(\mathbf{s};\mathbf{x})^{-1}\,\mathrm{d}\mathbf{x}\\ &= \int_{U(0)} + \sum_{1\leqslant\beta<\alpha}\int_{U_1(\alpha,\beta)} + \sum_{1\leqslant\alpha}\int_{U_1(\alpha)} + \sum_{1\leqslant\alpha}\int_{U(\alpha)}.\end{aligned}$$

We compute these sums separately. The integral over $U(0)$ is equal to 1. Then

$$\begin{aligned}
\sum_{1\leqslant\beta<\alpha}\int_{U_1(\alpha,\beta)} &= \frac{p-1}{p}\tau_p(f)\sum_{1\leqslant\beta<\alpha}p^{-\alpha s_0}p^{-\beta(s_1-s_0)}p^{\alpha n-\beta}\\
&= \frac{p-1}{p}\tau_p(f)\sum_{\beta=1}^{\infty}p^{-\beta(s_1-s_0+1)}\sum_{\alpha=\beta+1}^{\infty}p^{-\alpha(s_0-n)}\\
&= \frac{p-1}{p}\tau_p(f)\sum_{\beta=1}^{\infty}p^{-\beta(s_1-s_0+1)}p^{-\beta(s_0-n)}\frac{1}{p^{s_0-n}-1}\\
&= \frac{p-1}{p}\tau_p(f)\frac{1}{p^{s_0-n}-1}\sum_{\beta=1}^{\infty}p^{-\beta(s_1-n+1)}\\
&= \frac{p-1}{p}\tau_p(f)\frac{1}{p^{s_0-n}-1}\frac{1}{p^{s_1-n+1}-1}.
\end{aligned}$$

Concerning the integrals over $U_1(\alpha)$, we have

$$\sum_{1\leqslant\alpha}\int_{U_1(\alpha)} = \tau_p(f)\sum_{\alpha=1}^{\infty}p^{-\alpha s_1}p^{(n-1)\alpha} = \tau_p(f)\frac{1}{p^{s_1-n+1}-1}.$$

Finally,

$$\begin{aligned}
\sum_{1\leqslant\alpha}\int_{U(\alpha)} &= (1-p^{-n}-p^{-1}\tau_p(f))\sum_{\alpha=1}^{\infty}p^{-s_0\alpha}p^{n\alpha}\\
&= \big(1-p^{-n}-p^{-1}\tau_p(f)\big)\frac{1}{p^{s_0-n}-1}.
\end{aligned}$$

Adding all these terms gives

$$\begin{aligned}
\hat{H}_p(\mathbf{s};\psi_0) &= 1+(1-p^{-1})\tau_p(f)\frac{1}{p^{s_0-n}-1}\frac{1}{p^{s_1-n+1}-1}\\
&\qquad + \tau_p(f)\frac{1}{p^{s_1-n+1}-1} + \big(1-p^{-n}-p^{-1}\tau_p(f)\big)\frac{1}{p^{s_0-n}-1}\\
&= 1+(1-p^{-n})\frac{1}{p^{s_0-n}-1}\\
&\qquad + \tau_p(f)\Bigg((1-p^{-1})\frac{1}{p^{s_0-n}-1}\frac{1}{p^{s_1-n+1}-1}+\frac{1}{p^{s_1-n+1}-1}
\end{aligned}$$

$$
\begin{aligned}
&\qquad - p^{-1}\frac{1}{p^{s_0-n}-1}\Big)\\
&= \hat{H}_{\mathbf{P}^n,p}(s_0) + p^{-1}\tau_p(f)\frac{p-1+p^{s_0-n+1}-p-p^{s_1-n+1}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)}\\
&= \hat{H}_{\mathbf{P}^n,p}(s_0) + p^{-1}\tau_p(f)\frac{p^{s_0-n+1}-p^{s_1-n+1}}{(p^{s_0-n}-1)(p^{s_1-n+1}-1)}\\
&= \hat{H}_{\mathbf{P}^n,p}(s_0) + \tau_p(f)p^{n-1}\frac{p^{-s_1}-p^{-s_0}}{(1-p^{n-s_0})(1-p^{n-1-s_1})}
\end{aligned}
$$

□

§ 3. The local Fourier transform at a non-trivial character

In this subsection we evaluate the local Fourier transform at p for a non-trivial character $\psi_{\mathbf{a}}$. Let $S(\mathbf{a})$ be the union of S and of the set of primes p such that $\mathbf{a} \in p\mathbf{Z}^n$; We assume that $p \notin S(\mathbf{a})$.

Recall that $Z_f \subset \mathbf{P}^{n-1}_{\mathbf{Z}}$ denotes the subscheme defined by f and define $Z_{f,\mathbf{a}} = Z_f \cap H_{\mathbf{a}}$, where $H_{\mathbf{a}}$ is the hyperplane of $\mathbf{P}^{n-1}$ defined by $\mathbf{a}$. Finally, let $Z^t_{f,\mathbf{a}}$ (resp. $Z^{nt}_{f,\mathbf{a}}$) be the locus of points in $Z_{f,\mathbf{a}}$ where the intersection $Z_f \cap H_{\mathbf{a}}$ is *transverse* (resp. is *not transverse*). By assumption, Z_f and $H_{\mathbf{a}}$ are smooth over $\mathbf{Z}_p$.

Let $I(\alpha,\beta)$ be the integral of $\psi_{\mathbf{a}}$ over the set of $\mathbf{x} \in \mathbf{Q}_p^n$ such that $\|\mathbf{x}\| = p^\alpha$ and $|f(\mathbf{x})| \leqslant p^{d\alpha-\beta}$. Then, according to our partition of $\mathbf{Q}_p^n$, we have

$$
\begin{aligned}
\hat{H}_p(\mathbf{s};\psi_{\mathbf{a}}) &= 1 + \sum_{\alpha=1}^{\infty}\sum_{\beta=0}^{\alpha-1} p^{-\alpha s_0}p^{-\beta(s_1-s_0)}\int_{\substack{\|\mathbf{x}\|=p^\alpha\\ |f(\mathbf{x})|=p^{d\alpha-\beta}}}\psi_{\mathbf{a}}\\
&\qquad\qquad + \sum_{\alpha=1}^{\infty} p^{-\alpha s_1}\int_{\substack{\|\mathbf{x}\|=p^\alpha\\ |f(\mathbf{x})|\leqslant p^{-\alpha(d-1)}}}\psi_{\mathbf{a}}\\
&= 1 + \sum_{\alpha=1}^{\infty}\sum_{\beta=0}^{\alpha-1} p^{-\alpha s_0}p^{-\beta(s_1-s_0)}\big(I(\alpha,\beta)-I(\alpha,\beta+1)\big)\\
&\qquad\qquad + \sum_{\alpha=1}^{\infty} p^{-\alpha s_1} I(\alpha,\alpha).
\end{aligned}
$$

$$= 1 + \sum_{\alpha=1}^{\infty} p^{-\alpha s_0} I(\alpha,0) - (p^{s_1-s_0}-1) \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\alpha} p^{-\alpha s_0} p^{-\beta(s_1-s_0)} I(\alpha,\beta)$$

Lemma 3.1. — *If $t\in \mathbf{Q}_p$, the mean value over $\mathbf{Z}_p^*$ of $\psi(t\cdot)$ is equal to*

$$\frac{\int_{\mathbf{Z}_p^*}\psi(tu)\,du}{\int_{\mathbf{Z}_p^*}du} = \begin{cases} 1 & \text{if } t\in\mathbf{Z}_p;\\ -1/(p-1) & \text{if } v_p(t)=-1;\\ 0 & \text{if } v_p(t)\leqslant -2.\end{cases}$$

Proof. — Indeed, we have

$$\begin{aligned}\int_{\mathbf{Z}_p^*}\psi(tu)\,du &= \int_{\mathbf{Z}_p}\psi(tu)\,du - \int_{p\mathbf{Z}_p}\psi(tu)\,du\\ &= \int_{\mathbf{Z}_p}\psi(tu)\,du - \frac{1}{p}\int_{\mathbf{Z}_p}\psi(ptu)\,du.\end{aligned}$$

The integral of a non-trivial character over a compact group is 0, hence this integral equals 0 if $t\notin p^{-1}\mathbf{Z}_p$, equals $-\frac{1}{p}$ if $t\in p^{-1}\mathbf{Z}_p\setminus\mathbf{Z}_p$ and equals $1-\frac{1}{p}$ if $t\in\mathbf{Z}_p$. This proves the lemma. □

Using the change of variables $\mathbf{x}=p^{-\alpha}\mathbf{y}$, this implies the following formula:

$$(3.2)\quad I(\alpha,\beta) = p^{n\alpha}\Bigg(\frac{p}{p-1}\operatorname{vol}\big(\,\|x\|=1;\ p^\beta|f(\mathbf{x});\ p^\alpha|\langle\mathbf{a},\mathbf{x}\rangle\big) \\ -\frac{1}{p-1}\operatorname{vol}\big(\,\|x\|=1;\ p^\beta|f(\mathbf{x});\ p^{\alpha-1}|\langle\mathbf{a},\mathbf{x}\rangle\big)\Bigg).$$

Lemma 3.3. — *If $1\leqslant\beta\leqslant\alpha$, one has*

$$\operatorname{vol}\big(\,\|\mathbf{x}\|=1;\ p^\beta|f(\mathbf{x});\ p^\alpha|\langle\mathbf{a},\mathbf{x}\rangle\big) = p^{-\alpha}p^{(2-n)\beta}\big(1-\frac{1}{p}\big)\#Z_{f,\mathbf{a}}(\mathbf{Z}/p^\beta\mathbf{Z}).$$

In particular,

$$(3.4)\qquad I(\alpha,\beta)=0\quad\text{if } 1\leqslant\beta<\alpha.$$

Moreover, if $\alpha\geqslant 2$,

$$\begin{aligned}\operatorname{vol}\big(\,\|\mathbf{x}\|=1;\ p^\alpha|f(\mathbf{x});\ p^{\alpha-1}|\langle\mathbf{a},\mathbf{x}\rangle\big) &= \frac{1}{p}\operatorname{vol}\big(\,\|\mathbf{x}\|=1;\ p^{\alpha-1}|f(\mathbf{x});\ p^{\alpha-1}|\langle\mathbf{a},\mathbf{x}\rangle\big)\\ &= \frac{1}{p}p^{(1-\alpha)(n-1)}\big(1-\frac{1}{p}\big)\#Z_{f,\mathbf{a}}(\mathbf{Z}/p^{\alpha-1}).\end{aligned}$$

If $\alpha = 1$, one has

$$\operatorname{vol}\big(\,\|\mathbf{x}\| = 1;\, p | f(\mathbf{x})\big) = \big(1 - \frac{1}{p}\big) p^{1-n} \# Z_f(\mathbf{Z}/p\mathbf{Z}).$$

We had computed in [**5**], proof of Lemma 3.5, the integral

$$\int_{\|\mathbf{x}\|=p^\alpha} \psi_{\mathbf{a}} = \begin{cases} -1 & \text{if } \alpha = 1; \\ 0 & \text{if } \alpha \geqslant 2. \end{cases}$$

so that

$$\hat{H}(\mathbf{s};\psi_{\mathbf{a}}) = 1 - p^{-s_0} + \frac{p^{s_1-s_0}-1}{p-1} p^{-s_1} \# Z_f(\mathbf{F}_p) - \frac{p^{s_1-s_0}-1}{p-1}(1-p^{n-s_1-2}) \sum_{\alpha=1}^{\infty} p^{-\alpha(s_1-1)} \# Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}). \tag{3.5}$$

Lemma 3.6. — *For all $\alpha \geqslant 1$,*

$$\# Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha\mathbf{Z}) \leqslant p^{(n-3)(\alpha-1)} \# Z^t_{f,\mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) + p^{(n-2)(\alpha-1)} \# Z^{nt}_{f,\mathbf{a}}(\mathbf{Z}/p\mathbf{Z}).$$

Proof. — The inequality is trivially true for $\alpha = 1$. We prove it for any α by induction: to lift a point in $Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha\mathbf{Z})$ to a point in $Z_{f,\mathbf{a}}(\mathbf{Z}/p^{\alpha+1}\mathbf{Z})$, one needs to solve two equations in $\mathbf{u} \in \mathbf{F}_p^n$:

$$\langle \nabla f(\mathbf{x}), \mathbf{u}\rangle \equiv p^{-\alpha} f(\mathbf{x}), \quad \langle \mathbf{a}, \mathbf{u}\rangle \equiv p^{-\alpha}\langle \mathbf{a}, \mathbf{x}\rangle \pmod{p}.$$

A point in $Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha\mathbf{Z})$ which reduces to a point in $Z^t_{f,\mathbf{a}}$ modulo p has p^{n-3} lifts in $Z_{f,\mathbf{a}}(\mathbf{Z}/p^{\alpha+1}\mathbf{Z})$. On the other hand, a point reducing to a point in $Z^{nt}_{f,\mathbf{a}}$ has p^{n-2} or 0 lifts according to the two linear equations being compatible or not. This implies the lemma. □

Proposition 3.7. — *If not empty, the set $Z^{nt}_{f,\mathbf{a}}$ is a closed subscheme of bounded degree of $Z_{f,\mathbf{a}}$ and of dimension 0. There exist a constant C, independent of* $\mathbf{a}$ *and p such that*

$$\# Z^t_{f,\mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) \leqslant C p^{n-3}, \quad \# Z^{nt}_{f,\mathbf{a}}(\mathbf{Z}/p\mathbf{Z}) \leqslant C.$$

As a corollary, one gets:

Corollary 3.8. — *There exist a constant C such that for all α and $p \notin S(\mathbf{a})$,*

$$\# Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha\mathbf{Z}) \leqslant C p^{(n-3)\alpha} + C p^{(n-2)(\alpha-1)}.$$

Proof of Prop. 3.7. — The set $Z_{f,\mathbf{a}}$ is defined by the two equations $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x}\rangle = 0$. Fix the coordinates $x_1, \dots, x_n$ so that $\mathbf{a}$ is the first vector. Up to a constant, one may write

$$f(\mathbf{x}) = x_1^d + g_1(x_2, \dots, x_n)x_1^{d-1} + \cdots + g_{d_1}x_1 + g_d(x_2, \dots, x_n)$$

for some homogeneous polynomials g_i of degree i. Then, denoting $\mathbf{x} = (x_1, \mathbf{x}')$, $Z_{f,\mathbf{a}}$ is defined by the equations

$$x_1 = g_d(\mathbf{x}') = \partial_2 g_d(\mathbf{x}') = \cdots = \partial_n g_d(\mathbf{x}') = 0.$$

On $Z_{f,\mathbf{a}}$, $\partial_1 f(0, \mathbf{x}') = g_{d-1}(\mathbf{x}')$ and on $Z_{f,\mathbf{a}}^{nt} \subset Z_{f,\mathbf{a}}$, $\partial_i f(0, \mathbf{x}') = \partial_i g_d(\mathbf{x}')$. As Z_f is smooth, $g_{d-1}(\mathbf{x}')$ doesn't vanish on $Z_{f,\mathbf{a}}^{nt}$ which must therefore be either empty or of dimension 0. Its degree cannot exceed $d(d-1)^{n-1}$. The bound on the number of $\mathbf{F}_p$-rational points are a consequence of the following (certainly well-known) easy lemma. □

Lemma 3.9. — *Let $k = \mathbf{F}_q$ be a finite field, X a closed subscheme of $\mathbf{P}^n_k$ of dimension d. Then*

$$\#X(\mathbf{F}_q) \leqslant \mathbf{P}^d(\mathbf{F}_q) \deg X.$$

Proof. — We prove this by induction on d. If $d = 0$, the result is clear. Then, one can assume that X is reduced, irreducible and not contained in any hyperplane. For any hyperplane $H \subset \mathbf{P}^n$ which is rational over k, $X \cap H$ is a closed subscheme of H of dimension $d - 1$ and of degree $\leqslant \deg X$. By induction, we have

$$\#(X \cap H)(\mathbf{F}_q) \leqslant \#\mathbf{P}^{d-1}(\mathbf{F}_q) \deg X.$$

Finally, any point of $X(\mathbf{F}_q)$ is contained in exactly $\#\mathbf{P}^{n-1}(\mathbf{F}_q)$ rational hyperplanes in $\mathbf{P}^n$, so that

$$\#X(\mathbf{F}_q)\#\mathbf{P}^{n-1}(\mathbf{F}_q) \leqslant \#\mathbf{P}^{d-1}(\mathbf{F}_q)\#\mathbf{P}^n(\mathbf{F}_q) \deg X.$$

As $n \geqslant d$, this implies

$$\#X(\mathbf{F}_q) \leqslant \frac{q^{n+1}-1}{q^n-1}\frac{q^d-1}{q-1} \deg X \leqslant \mathbf{P}^d(\mathbf{F}_q) \deg X.$$

□

§ 4. The height zeta function

From now on, we fix some $\varepsilon > 0$ and consider only $\mathbf{s}$ in the subset Ω of $\mathbf{C}^2$ defined by the inequalities $\mathrm{Re}(s_0) > n + \varepsilon$ and $\mathrm{Re}(s_1) > n - 1 + \varepsilon$.

Proposition 4.1. — *There exist a holomorphic function g on Ω which has polynomial growth in vertical strips such that*

$$\hat{H}(\mathbf{s}, \psi_0) = g(\mathbf{s}) \frac{1}{(s_0 - n - 1)(s_1 - n)}.$$

Proof. — Indeed, we see from 2.3 that for $p \notin S$,

$$\hat{H}_p(\mathbf{s}, \psi_0) = 1 + p^{n-s_0} + p^{n-s_1-1} + O(p^{-1-\varepsilon}),$$

the O being uniform in p. Consequently,

$$\prod_{p \notin S} \hat{H}_p(\mathbf{s}, \psi_0)(1 - p^{n-s_0})(1 - p^{n-1-s_1})$$

converges to a holomorphic bounded function on Ω. As the finite number of remaining factors converge uniformly in Ω, the existence of g is proven. The growth of g in vertical strips follows from Rademacher's estimates for the Riemann zeta function. □

Lemma 4.2. — *There exist a constant $C > 0$ such that for all $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$, all $p \notin S(\mathbf{a})$ and all $(s_0, s_1) \in \Omega$, one has*

$$\left| \hat{H}_p(\mathbf{s}, \psi_{\mathbf{a}}) - 1 \right| \leqslant C p^{-1-\varepsilon}.$$

Proof. — Recall the formula 3.5:

$$\hat{H}_p(\mathbf{s}, \psi_{\mathbf{a}}) - 1 = -p^{-s_0} + \frac{p^{-s_0} - p^{-s_1}}{p-1} p^{n-2} (1 - \frac{1}{p})^{-1} \tau_p(f) - \frac{p^{s_1 - s_0} - 1}{p - 1} (1 - p^{n-s_1-2}) \sum_{\alpha=1}^{\infty} p^{-\alpha(s_1-1)} \# Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z})$$

the right hand side of which we have to estimate all terms. The first one is $p^{-s_0} = O(p^{-1-\varepsilon})$. Then, as $\tau_p(f)$ is bounded, the second one is

$$O(p^{n-3-\mathrm{Re}(s_0)}) + O(p^{n-3-\mathrm{Re}(s_1)}) = O(p^{-2}).$$

For the last term T_3, we use Lemma 3.8 so that, denoting $\sigma_1 = \mathrm{Re}(s_1)$,

$$\begin{aligned}\sum_{\alpha=1}^{\infty} p^{-\alpha(s_1-1)} \# Z_{f,\mathbf{a}}(\mathbf{Z}/p^\alpha \mathbf{Z}) \\ &\leqslant C\sum_{\alpha=1}^{\infty} p^{-\alpha(\sigma_1-1)} p^{(n-3)\alpha} + C\sum_{\alpha=1}^{\infty} p^{-\alpha(\sigma_1-1)} p^{(n-2)(\alpha-1)} \\ &\leqslant C\frac{1}{p^{\sigma_1-n+2}-1} + Cp^{2-n}\frac{1}{p^{\sigma_1-n+1}-1}.\end{aligned}$$

Moreover,

$$\left|1-p^{n-s_1-2}\right| \leqslant 2$$

so that

$$\begin{aligned}|T_3| &\ll \frac{1}{p-1}\frac{p^{\sigma_1-\sigma_0}+1}{p^{\sigma_1-n+2}-1} + 2C\frac{p^{2-n}}{p-1}\frac{p^{\sigma_1-\sigma_0}+1}{p^{\sigma_1-n+1}-1} \\ &\ll \frac{1}{p}\left(p^{n-2-\sigma_0}+p^{n-2-\sigma_1}\right) + p^{1-n}\left(p^{n-1-\sigma_0}+p^{n-1-\sigma_1}\right) \\ &\ll p^{-2}.\end{aligned}$$

The lemma is proved. □

Proposition 4.3. — *For each* $\mathbf{a} \in \mathbf{Z}^n \setminus \{0\}$, $\hat{H}(\mathbf{s}, \psi_{\mathbf{a}})$ *is a holomorphic function on* Ω. *Moreover, there exist constants* $C > 0$ *and* ν *(which are independent of* $\mathbf{s}$ *and* $\mathbf{a}$*) such that*

$$\left|\hat{H}(\mathbf{s}, \psi_{\mathbf{a}})\right| \leqslant C(1+\|\Im(s)\|)^{\nu}(1+\|\mathbf{a}\|)^{-n-1}.$$

Proof. — Write

$$\hat{H}(\mathbf{s}, \psi_{\mathbf{a}}) = \prod_{p\notin S(\mathbf{a})} \hat{H}_p \times \prod_{p\in S(\mathbf{a})} \hat{H}_p \times \hat{H}_\infty.$$

The convergence of the first infinite product to a bounded holomorphic function follows from the preceding lemma. As in Lemma 3.7 of [**5**], there exists a constant $\kappa > 0$ such that

$$\left|\prod_{p\in S(\mathbf{a})} \hat{H}_p(\mathbf{s}, \psi_{\mathbf{a}})\right| \ll (1+\|\mathbf{a}\|)^{\kappa}.$$

Using the rapidly decreasing behaviour of $\hat{H}_\infty$ as a function of $\mathbf{a}$

$$\left|\hat{H}_\infty(\mathbf{s},\psi_{\mathbf{a}})\right| \ll (1+\|\mathbf{a}\|)^{-n-\kappa-1}$$

established in Prop. 2.13 of *loc. cit.*, the proposition is proved. □

Theorem 4.4. — *The height zeta function converges in the domain* $\mathrm{Re}(s_0) > n+1$, $\mathrm{Re}(s_1) > n$. *Moreover, there exists a holomorphic function* g *in the domain* $\mathrm{Re}(s_0) > n$, $\mathrm{Re}(s_1) > n-1$ *such that*

$$Z(\mathbf{s}) = g(\mathbf{s})\frac{1}{(s_0-n-1)(s_1-n)}.$$

The function g *has polynomial growth in vertical strips and* $g(n+1,n)\neq 0$.

Specializing to $\mathbf{s} = s(n+1,n)$ and using a standard Tauberian theorem, one obtains the following corollary.

Corollary 4.5. — *There exist a polynomial* P_X *of degree* 1 *and a real number* $\alpha > 0$ *such that the number of points of* $U(\mathbf{Q}) \subset X(\mathbf{Q})$ *of anticanonical height* $\leqslant B$ *satisfies*

$$N(U, K_X^{-1}, B) = BP(\log B) + O(B^{1-\alpha}).$$

Moreover, if $\tau(K_X)$ *denotes the Tamagawa number, the leading coefficient of* P_X *is equal to*

$$\frac{\tau(K_X)}{(n+1)n},$$

as predicted by Peyre's refinement of Manin's conjecture.

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